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Generation of Random Hypersurfaces

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A technique to generate multidimensional random hypersurfaces with given statistical properties numerically on a digital computer is discussed in this paper. The technique is based on the linear filter theory. Using this technique, two-dimensional random surfaces are generated to illustrate the procedure.

Statistics of these two-dimensional random surfaces are presented. Possible applications of these random surfaces in various physical problems are discussed.

1. INTRODUCTION

The study of multidimensional random functions is of importance both in theory and in practice. It is a study of interest not only in mathematics, but also in applications of a wide range of physical problems. To name a few examples, we have light scattering from ocean surfaces [1], wave propagation in turbulent media [2], heating and diffusion in velocity space of plasmas in stochastic fields [3], planetary motions in random gravitational fields [4], and random light rays [5]. In most of these problems, analytic computations usually do not go very far. It is then necessary to resort to some type of Monte Carlo calculations in an attempt to simulate the physical situation. The solution to one class of such problems can be approached by first generating random hypersurfaces with prescribed statistical properties. For example, the study of random light rays can be carried out by first generating random refractive index surfaces and then tracing rays in them to obtain statistical properties. Other examples given above can be done similarly or with some variations.

Many techniques are available to generate independent random numbers distributed uniformly over an interval. To obtain random surfaces with prescribed correlations digitally, we need to assemble these numbers and pass them through a filter in a proper manner. The filter must have the desired characteristics so that its output has the desired statistical properties. The paper starts in Section 2 with a quick review of the filtering theory which is then followed by descriptions of a procedure by which the filter can be synthesized. As a concrete example, two-dimensional random surfaces with a prescribed correlation function are generated in Section 3. Statistical properties of the generated random surfaces are investigated in Section 4.

2. Generation of Random Functions

Let us consider a noncausal, linear, and shift-invariant N-dimensional filter. Let $\mathbf{P}$ and $\mathbf{Q}$ be points in the N-dimensional space. The input $X(\mathbf{P})$ and the output $Y(\mathbf{Q})$ of the filter are related through the impulse response $h(\mathbf{P}, \mathbf{Q}) = h(\mathbf{Q} - \mathbf{P})$ by the convolution integral. When the input $X(\mathbf{P})$ is a real homogeneous random process with correlation function $R_{XX}(\mathbf{P}_1 - \mathbf{P}_2)$ and spectral density function $S_{XX}(\mathbf{k})$, the output $Y(\mathbf{Q})$ must also be a real homogeneous random process. Let the output correlation function and spectral density function be, respectively, $R_{YY}(\mathbf{Q}_1 - \mathbf{Q}_2)$ and $S_{YY}(\mathbf{k})$. Then the output is related to input by

$$S_{YY}(\mathbf{k}) = S_{XX}(\mathbf{k}) |H(\mathbf{k})|^2, \tag{1}$$

where $H(\mathbf{k})$ is the Fourier transform of the impulse response h. In the following we will call h the correlator function for obvious reasons. Equation (1) is just the well-known relation in filtering theory and system analysis. We will use it to synthesize the filter so that when a known input process is fed into the filter, the output process has the desired correlation function. To do this digitally on a computer requires us to extend the analysis to the discrete case. The input at the grid point $\mathbf{p} = (p_1, p_2, ..., p_N)$ can be written as

$$X(\mathbf{p}) = \sum_{i_1=-\infty}^{\infty} \sum_{i_2=-\infty}^{\infty} \cdots \sum_{i_N=-\infty}^{\infty} B(\mathbf{i})\, \delta_{\mathbf{ip}}, \tag{2}$$

where B is the input process with known distribution and correlation. Because of the presence of Kronecker delta function $\delta_{\mathbf{ip}}$, the input (2) takes the value of B at the discrete mesh points at which $\mathbf{i} = \mathbf{p}$. The output of the discrete system is then given by

$$Y(\mathbf{q}) = \sum_{j_1=-\infty}^{\infty} \sum_{j_2=-\infty}^{\infty} \cdots \sum_{j_N=-\infty}^{\infty} X(\mathbf{j})\, h(\mathbf{q} - \mathbf{j}), \tag{3}$$

where $\mathbf{q} - \mathbf{j} = (q_1 - j_1, q_2 - j_2, ..., q_N - j_N)$.

Substituting (2) into (3), we obtain

$$Y(\mathbf{q}) = \sum_{i_1=-\infty}^{\infty} \sum_{i_2=-\infty}^{\infty} \cdots \sum_{i_N=-\infty}^{\infty} B(\mathbf{i})\, h(\mathbf{q} - \mathbf{i}). \tag{4}$$

The correlation coefficient for the output in the discrete system can be obtained from (4) to give

$$R_{YY}(\boldsymbol{\xi}) = \sum_{\mathbf{i}} \sum_{\mathbf{j}} R_{XX}(\mathbf{i} - \mathbf{j})\, h(-\mathbf{i})\, h^*(\boldsymbol{\xi} - \mathbf{j}), \tag{5}$$

where the $\sum_{\mathbf{i}}$ stands for multiple summation for all values of $\mathbf{i} = (i_1, i_2, ..., i_N)$ and similarly for $\sum_{\mathbf{j}}$. The input correlation coefficient in (5) is

$$R_{XX}(\mathbf{i} - \mathbf{j}) = \langle B(\mathbf{i})\, B(\mathbf{j})\rangle. \tag{6}$$

For the special case for which $X(\mathbf{p})$'s are independent random variables, (5) reduces to

$$R_{YY}(\boldsymbol{\xi}) = \langle B^2\rangle \sum_{\mathbf{i}} h(\mathbf{i})\, h(\boldsymbol{\xi} - \mathbf{i}), \tag{7}$$

where $\langle B^2\rangle$ is now a constant.

For a prescribed $R_{YY}(\boldsymbol{\xi})$, (7) may be used to solve for the correlator function h and then (3) used to find the required output $Y(\mathbf{q})$. In practice, however, the number of random variables that can be generated on a digital computer is limited, the summation in (3) and (7) have to be truncated. Let the truncation be confined to the N-dimensional square with $2L$ on each side. Then (3) and (7) become, respectively,

$$Y(\mathbf{q}) = \sum_{\mathbf{i}=\mathbf{q}-\mathbf{L}}^{\mathbf{q}+\mathbf{L}} B(\mathbf{i})\, h(\mathbf{q} - \mathbf{i}), \tag{8}$$

and

$$R_{YY}(\boldsymbol{\xi}) = \langle B^2\rangle \sum_{\mathbf{i}=0}^{2\mathbf{L}} h(\mathbf{L} - \mathbf{i})\, h(\mathbf{L} + \boldsymbol{\xi} - \mathbf{i}). \tag{9}$$

To solve for h by using (9) is an inversion problem. In practice, however, even for the two-dimensional case with moderate values of $\mathbf{L}(L_1 = L_2 = \cdots = 5$, for instance), (9) represents a set of 121 simultaneous quadratic equations. For general N-dimensional hypersurfaces, the number of quadratic equations is given by $(2L + 1)^N$. The fact that the equations are quadratic makes the inversion problem rather time consuming on a computer. Therefore, instead of using (8) and (9) directly, we have adopted a mixed continuum-discrete procedure in our computations. This hybrid procedure begins with the given output correlation function from which $H(\mathbf{k})$ and, consequently, $h(\mathbf{P})$ can be found by using (1). The continuous correlator function is then evaluated at the discrete points in a manner given by (8) to yield the output $Y(\mathbf{q})$ at each discrete grid point.

In the following section a two-dimensional example is given to show the details of this procedure.

3. Random Surfaces

As an example for the application of the procedure described in the last section, we now generate two-dimensional random surfaces with the prescribed correlation function given by

$$R_{YY}(\tau_1, \tau_2) = A[(\tau_1/l_1)^2 + (\tau_2/l_2)^2]\, K_2\{[(\tau_1/l_1)^2 + (\tau_2/l_2)^2]^{1/2}\}, \tag{10}$$

where A is a normalization constant, $K_2(\rho)$ is the modified Bessel's function of the second kind, and l_1, l_2 are proportional to the correlation lengths in the two orthogonal directions on the surface, say x and y, respectively. The correlation function of the form of (10) is of interest in the study of wave propagation through atmospheric turbulence [2]. The corresponding power spectrum for (10) is

$$S_{YY}(k_1, k_2) = A_1/\{[1 + (k_1 l_1)^2 + (k_2 l_2)^2]^3\}. \tag{11}$$

Our procedure begins with the generation of independent, uniformly distributed random numbers using the multiplicative congruential method [6]. Figure 1 shows a histogram of 13225 uniformly distributed and independent random numbers (X_i) generated on the interval (0, 1) by this method. Using the transformation

$$\begin{aligned} Y_i &= (-2 \ln X_i)^{1/2} \cos 2\pi X_{i+1}, \\ Y_{i+1} &= (-2 \ln X_i)^{1/2} \sin 2\pi X_{i+1}, \end{aligned} \tag{12}$$

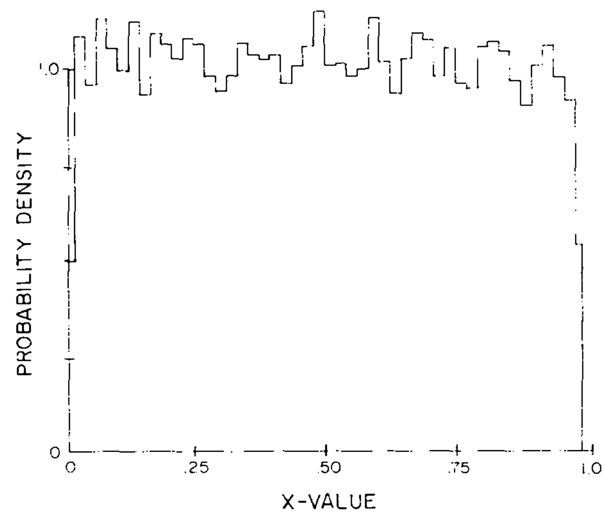

FIG. 1. Histogram of 13225 uniformly distributed and independent random numbers generated on the interval (0, 1).

we obtain the same number of Gaussian-distributed independent random numbers (Y_i). These numbers are then arbitrarily arranged in the form of a two-dimensional array and serve as the input to our correlator. Since the input signal consists of independent random numbers, the continuous correlator function may be found by using (1) and (11). The resulting correlator function is

$$h(\tau_1, \tau_2) = A_2 \exp\{-[(\tau_1/l_1)^2 + (\tau_2/l_2)^2]^{1/2}\}, \tag{13}$$

where A_2 is a constant.

To apply this correlator function to the discrete case, we write

$$h(i_1, i_2) = A_2 \exp[-(\alpha_1^2 i_1^2 + \alpha_2^2 i_2^2)^{1/2}], \tag{14}$$

where α_1 and α_2 are parameters to be determined from the ratios of the distance between grid points to the correlation lengths.

Substituting (14) into (8) and (9), we obtain for this two-dimensional random surface

$$Y(q_1, q_2) = \sum_{i_1=q_1-L}^{q_1+L} \sum_{i_2=q_2-L}^{q_2+L} A_2 B(i_1, i_2) \exp\{-[(q_1 - i_1)^2 \alpha_1^2 + (q_2 - i_2)^2 \alpha_2^2]^{1/2}\} \tag{15}$$

$$R_{YY}(\xi_1, \xi_2) = A_2 \langle B^2 \rangle \sum_{i_1=0}^{2L} \sum_{i_2=0}^{2L} \exp\{-[(L - i_1)^2 \alpha_1^2 + (L - i_2)^2 \alpha_2^2]^{1/2}\} \cdot \exp\{-[(L + \xi_1 - i_1)^2 \alpha_1^2 + (L + \xi_2 - i_2)^2 \alpha_2^2]^{1/2}\}, \tag{16}$$

where $B(i_1, i_2)$ are the array of Gaussian-distributed independent random numbers.

Equation (15) gives the desired array of random numbers whose correlation function approximates the given correlation function (10).

To determine α_1 and α_2, we compare the continuous correlation function (10) with the discrete one (16). Let us take the distances between the grid points to be ζ_1 and ζ_2 along the two orthogonal axis. Assume the correlation lengths are given by $l_1 = n_1\zeta_1$ and $l_2 = n_2\zeta_2$, where n_1 and n_2 are integers.

In Fig. 2, cross-sectional plots of (10) are drawn as functions of $\tau_1 = p\zeta_1$ (p, integers) for $\tau_2 = 0$ at different values of n_1. On top of these plots, plots of (16) are also given for $\xi_2 = 0$ as a function of $\xi_1 = p\zeta_1$ (p, integers) at different values of α_1. An eleven-by-eleven matrix is used to compute each point in (16), corresponding to $L = 5$. From these plots, we see that for a given value n_1, we can choose a value α_1 such that the curve given by (16) approximates that given by (10). The actual fitting of the curves is done by a least-square computation. This way we can determine the parameter α_1 for a given correlation length. For isotropic surface, $n_1 = n_2$, and $\alpha_1 = \alpha_2$. For anisotropic surfaces, cross-sectional plots along $\tau_1 = 0$ and $\xi_1 = 0$ may be used to find the value of α_2 for a given n_2.

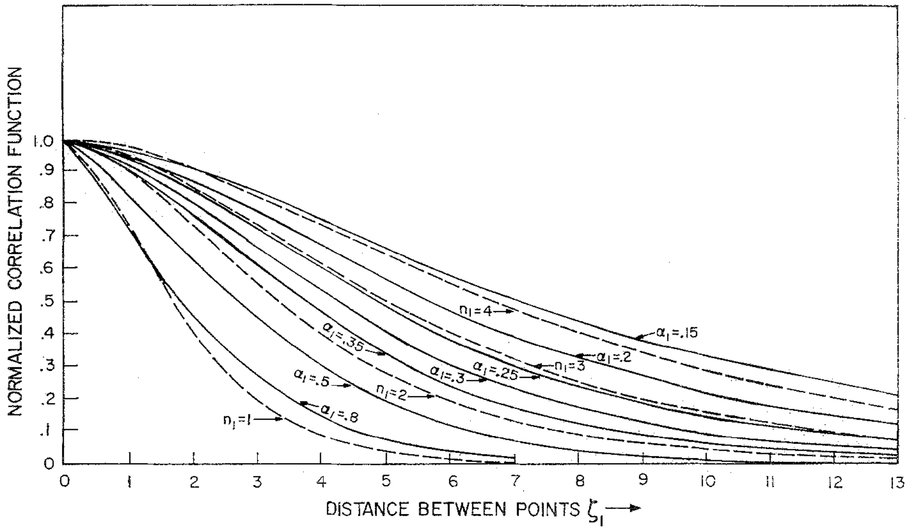

FIG. 2. Cross-sectional plots of Eq. (10) drawn as functions of $\tau_1 = p\zeta$ (p, integers) for $\tau_2 = 0$ at different values of n_1. Also, plots of Eq. (16) for $\xi_2 = 0$ as functions of $\xi_1 = p\zeta_1$ (p, integers) at different values of α_1.

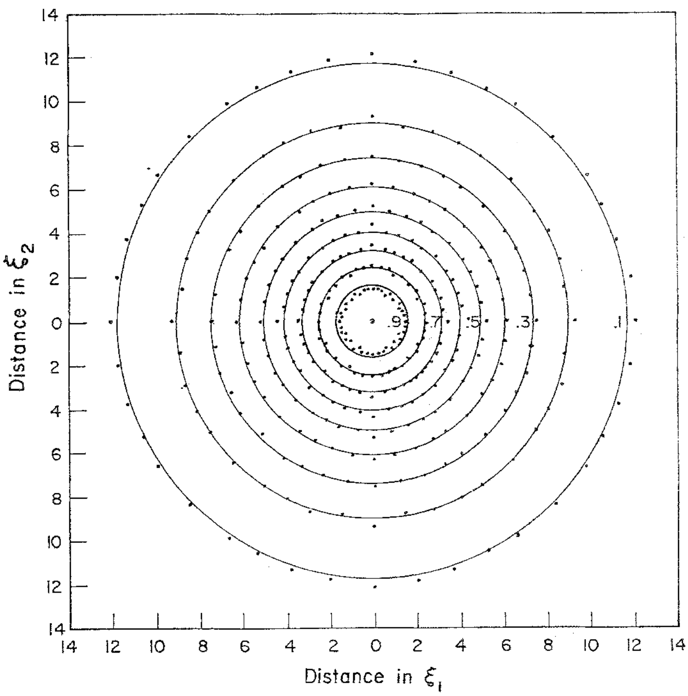

FIG. 3. Correlation function contours computed for Eq. (16) for $\alpha_1 = \alpha_2 = 0.25$ as compared with the contours of the continuous correlation function Eq. (10).

For the specific example of isotropic random surfaces we choose $n_1 = n_2 = 3$, corresponding to four grid points in one correlation length. The values of α_1 and α_2 are found to be 0.25. Using these values, the correlation contours computed from (16) are shown in Fig. 3 as compared with those of the continuous correlation function (10). We note that they are close to each other. The correlated two-dimensional arrays are then computed by substituting $\alpha_1 = \alpha_2 = 0.25$ into (15). Six such spaces are generated for six different input arrays of independent random numbers. The statistical properties of these surfaces are discussed in the next section.

The anisotropic random surfaces can be computed similarly by using unequal n_1 and n_2. Four such surfaces have been computed by using $n_1 = 1$ and $n_2 = 3$. The corresponding values are $\alpha_1 = 0.8$ and $\alpha_2 = 0.15$.

These surfaces all have 105 grid points on each side.

4. Statistics of the Random Surfaces

The distribution of the generated random numbers $Y(q_1, q_2)$ is shown in the histogram of Fig. 4 for one of the spaces. Two chi-square tests are made on these numbers and show that the distribution is Gaussian with mean $= 0$ and variance $= 1$ (for 100 cells in the histogram, $\chi^2 = 108.5$; for 48 cells in the histogram, $\chi^2 = 53.4$).

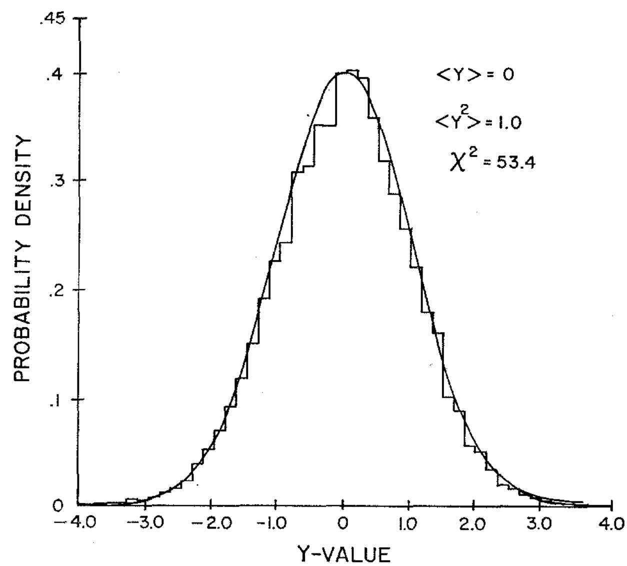

FIG. 4. Distribution of the generated random numbers $Y(q_1, q_2)$.

The correlation coefficients for the random surfaces are computed by the formula

$$R_{YY}(m_1\zeta, m_2\zeta) = \frac{1}{(N - m_1)(N - m_2)} \sum_{i_1=0}^{N-m_1} \sum_{i_2=0}^{N-m_2} \times Y(i_1\zeta, i_2\zeta)\, Y[(i_1 + m_1)\, \zeta, (i_2 + m_2)\, \zeta], \tag{17}$$

where N is the number of grid points on each side of the array. Figure 5 shows the computed correlation coefficient contours for one of the isotropic random surfaces as compared with the contours of the continuous correlation function (10). We note that the computed values are consistently less than those of the continuous correlation function. This feature was found in all the other cases computed. The discrepancy may be attributed to the following two causes. First, instead of using (9) directly, we have adopted the hybrid procedure, this certainly is to affect the accuracy. Second, the truncation of the series in (8) and (9) also introduces error. Computations made on a smaller space indicate that when the value of L is increased in (8), the computed correlation contours fit the continuous contours better. It is of interest to note from Fig. 5 that the generated space is quite isotropic. A cross section of the correlation contours (Fig. 6) shows that it fits very well the continuous correlation function (10) with a correlation length about 4/5 of the original one.

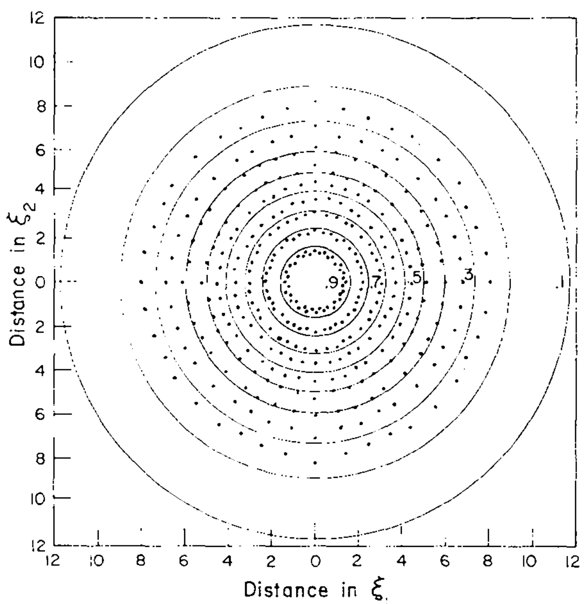

FIG. 5. The computed correlation coefficient contours for one of the isotropic random surfaces as compared with the contours of the continuous correlation function Eq. (10).

In Fig. 7 a contour plot of one of the generated isotropic surfaces is shown. It may be used to simulate the two-dimensional random fluctuations of the refractive index of a certain medium, or to represent the random height variations of a rough terrain.

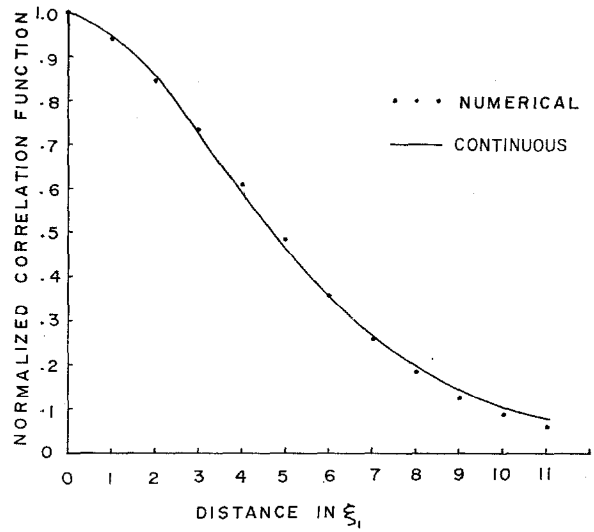

FIG. 6. Cross-sectional plot of the computed correlation coefficient as compared with the continuous correlation function Eq. (10) with correlation length 4/5 of the original one.

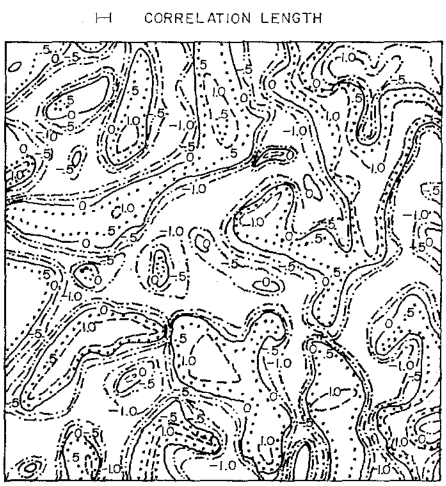

FIG. 7. Contour plot of the generated isotropic surfaces.

5. Conclusion

In this paper we have presented a rather straightforward procedure to generate N-dimensional random functions with given correlation functions. The procedure is demonstrated through the generation of two-dimensional random surfaces. A typical 105×105 surface takes between 3 to 4 minutes of computing time of IBM 360/75. Adams and Denman [7] have used a similar procedure to generate one-dimensional random functions. Rusbridge [8] has used a different approach to generate 25×25 random surfaces in his study of scattering of microwaves by a turbulent plasma.

Due to the possible errors caused by the hybrid procedure and the truncation in generating the space, the correlation length of the generated space is found to be consistently smaller than that of the original random function. This discrepancy may be significant in the computations for certain physical problems. Because of this, it is suggested that careful studies of the generated spaces such as those indicated on Figs. 5 and 6 should be carried out and the correct value of the correlation length be chosen before these spaces are used in the computation.

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